Optical phase shifts and diabolic topology in Möbius-type strips

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We compute the optical phase shifts between the left and the right-circularly polarized light after it traverses non-planar cyclic paths described by the boundary curves of closed twisted strips. The evolution of the electric field along the curved path of a light ray is described by the Fermi-Walker transport law which is mapped to a Schrödinger equation. The effective quantum Hamiltonian of the system has eigenvalues equal to $0, \pm\kappa$, where κ is the local curvature of the path. The inflexion points of the twisted strips correspond to the vanishing of the curvature and manifest themselves as the diabolic crossings of the quantum Hamiltonian. For the Möbius loops, the critical width where the diabolic geometry resides also corresponds to the characteristic width where the optical phase shift is minimal. In our detailed study of various twisted strips, this intriguing property singles out the M"obius geometry.

The geometrical phenomenon of anholonomy relates to the inability of a variable to return to its original value after a cyclic evolution.[1] An example of this phenomenon is the change in the direction of polarization of light in a coiled optical fiber.[2, 3] This leads to an optical phase shift which corresponds to a phase change between a right and a left-handed circularly polarized wave, when they travel along a non-planar path. [4] The effect is an optical manifestation of the Aharonov-Bohm phase, according to which two electron beams develop a phase shift proportional to the magnetic flux they enclose.[5] For the polarized light the analog of magnetic flux is the solid angle subtended on the sphere of directions $\hat{\mathbf{k}}$, where $\hat{\mathbf{k}}$ represents the direction of propagation of light, which changes as light passes through a twisted fiber. In previous studies involving the propagation of light through a helically wound optical fiber[2, 3, 4], the simple geometry of a circular helix gives this solid angle to be equal to $\Omega = 2\pi(1 - \cos\ \theta)$, where θ is the pitch angle of the helix, or the angle between the axis of the helix and the local axis of the optical fiber.

In this paper, we compute the optical phase shift between the left and the right-circularly polarized light as it passes through the optical fibers with a more complex geometry, namely, fibers shaped like the *boundary curves* of closed twisted strips. Although we study a whole class of twisted strips, our main focus is on the Möbius strip whose unique geometry has been a source of constant fascination. Here we explore the interplay between the intrinsic geometry of the strips and the geometrical physical phenomenon, namely the geometric phase shift experienced by light as it passes through the boundary curve of the strips. Interestingly, the Möbius geometry is singled out, since the optical phase shift is found to exhibit a characteristic minimum as the width of the strip is continuously varied.

The class of twisted, closed strips under consideration is described by,

$$
\begin{aligned}
x &= (1 + w\cos\frac{nt}{2})\cos t; \\
y &= (1 + w\cos\frac{nt}{2})\sin t; \\
z &= w\sin\frac{nt}{2},
\end{aligned}
$$
(1)

with parameters $-\alpha \leq w \leq \alpha$ and $0 \leq t \leq 2\pi$. Thus α is the half-width of the strip and its length $L = 2\pi$. The integer n is the number of half-twists on the strip (so that $n = 1$ refers to the well-known Möbius strip). We are interested in the geometry and topology of the boundary curve of the twisted strip. The parametric equation for this space curve is found by setting $w = \pm\alpha$ in Eq. (1). For odd n, the strip is a non-orientable surface with one boundary curve, so that the range of t is $[0, 4\pi]$. For even n, the strip is orientable, with two boundary curves with similar geometries, and the range of t is $[0, 2\pi]$ for each of these curves. Topologically, for all widths, the boundaries of the odd-n strips with $n > 1$ are knotted curves (e. g., it is a trefoil knot for $n = 3$, a five-pointed star knot for $n = 5$, etc.), while those of the even-n strips are not knotted. The boundary curve of the Möbius strip clearly does not fall in either of these classes, since it is the only case in which the boundary curve of a non-orientable strip is not a knot. Figure 1 shows the boundary curves of the Möbius strip for various values of α. One of the characteristic feature of the twisted strips are the inflexion points, distinguished by a local "straightening" around that point. Various details regarding such points for n=twisted strips and their non-trivial dependence on the width-to-length ratio will be discussed later. An important point of this paper is that for the Möbius strip, the inflexion point manifests itself in a rather unique way in encoding the polarization

properties of light as it traverses the Möbius loop.

We consider the propagation of circularly polarized light along an optical fiber which has the shape of the boundary curve of a twisted strip. The direction of propagation of light is tangential to the boundary curve and hence completely encodes the geometry of the boundary curve. As the light completes a full loop of the twisted strip boundary, its direction of propagation, the tangent indicatrix, completes a closed circuit on the surface of the sphere of directions defined by the propagation vector $\hat{\mathbf{k}}$. By adiabatically varying the direction of propagation around a closed circuit on the sphere, the change in the polarization of light is equal to the solid angle subtended in $\mathbf{k}$-space. Condition of adiabaticity is that the length of the boundary curve be large as compared to the wavelength of light.

The boundary curve of a twisted strip, viewed as a space curve $\Gamma : \mathbf{r}(s) = (x, y, z)$, thus represents the path along which light propagates. Here s is the arc length measured along the curve, with $ds = v\, dt$, where $v = |\frac{d\mathbf{r}}{dt}|$. The geometry of the space curve can be described by the right-handed orthonormal triad $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ that represent unit tangent, normal and binormal at every point along the trajectory. The evolution of the triad on the curve can be described by Frenet-Serret equations[10],

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}; \ \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}; \ \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}, \quad (2)$$

where $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. κ and τ denote, respectively, the local curvature and the torsion. They are given by[10]

$$\kappa = |\mathbf{T}(s)| \quad ; \quad \tau = [\mathbf{T} \cdot \frac{d\mathbf{T}}{ds} \times \frac{d^2\mathbf{T}}{ds^2}]/\kappa^2, \quad (3)$$

Intuitively, the curvature measures the deviation of the curve from a straight line, while the torsion quantifies the non-planarity of the curve.

In order to compute the change in the polarization of light as it passes through a circuit in the shape of the boundary of a twisted ribbon, we follow the variation in the electric field vector with respect to t (or s) as light travels along the curve Γ. Using Maxwell's equations, it can be shown that the evolution of the complex unit vector $\hat{\mathbf{E}}$ associated with the complex three-component electric field on this space curve follows the Fermi-Walker transport law[8, 9],

$$\frac{d\hat{\mathbf{E}}}{ds} = \kappa(s)\mathbf{B} \times \hat{\mathbf{E}} \quad (4)$$

We first note that interestingly, Eq. (4) can be cast in the form of a Schrödinger equation with the following Hamiltonian $H(s)$, which is a three-dimensional, antisymmetric Hermitian matrix, with pure imaginary elements.

$$H = i\kappa \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix} \quad (5)$$
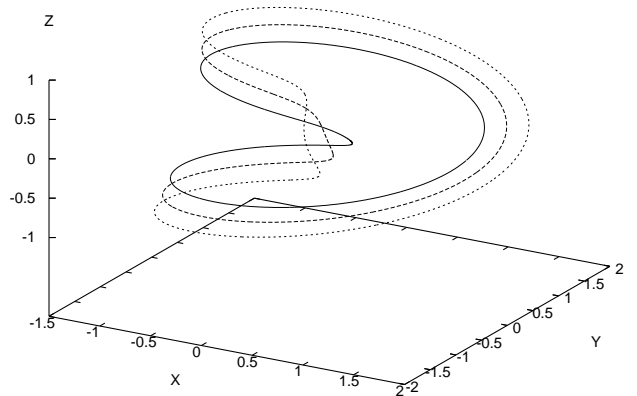

FIG. 1: Boundary curves of the closed Möbius strip (Eq. (1) with $n = 1$) for strip-widths $\alpha = 1.0, 0.8, 0.6$ (outermost to innermost). Note the local straightening around the inflexion point for the critical curve with $\alpha = 0.8$.

Here B_i is the ith-component of the binormal vector $\mathbf{B}$. A short calculation shows that the eigenvalues of this Hamiltonian are $0, \pm\kappa$. Hence the quantum Hamiltonian exhibits a 3-fold degeneracy at points where κ vanishes. These are just the inflexion points of the optical fiber, and this is a general result that follows from the basic evolution Eq. (4). For the example of a fiber shaped like the boundary curve of an n-twisted strip, there are n such inflexion points (see below) that appear when the width of the strip takes on a certain critical value, which depends on n.

The vanishing of κ implies that the quantum Hamiltonian is degenerate at the the critical point and as t and α vary on the surface of the strip, this degeneracy leads to a diabolic point. Fig. (2) illustrates the conical geometry near the critical point. Further details about such points will be discussed later in this paper. Such diabolic crossings in $t - \alpha$ space are sensed by a circuit that does not pass through the degeneracy, but simply encloses it. In particular, such closed loops result in wave functions that acquire a Berry phase that depends on the geometry of the path in *parameter space* in an adiabatic cyclic evolution. In contrast to Berry phase, which is obtained by a mathematical mapping of an optical system to a quantum problem, and further, is associated with parametric circuits enclosing a conical intersection, the geometric phase that we investigate involves circuits that are in *configuration space*, i.e., boundary curves of twisted strips. Hence, while a special circuit may pass through a diabolic crossing, typical circuits never enclose one. More importantly, this geometric phase can be directly measured in experiments, making it more interesting and applicable, as we shall discuss at the end.

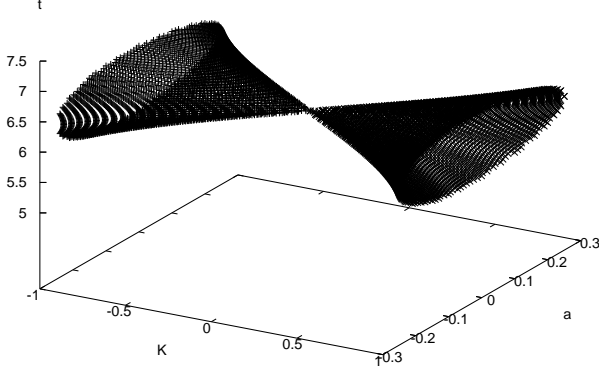The geometric phase is obtained by solving Eq. (4)

FIG. 2: Conical topology of the eigenvalues $\pm\kappa(t,\alpha)$ of the quantum Hamiltonian (Eq. (5)) near the critical point (t_c,α_c) on the surface of the Möbius strip. Here $a = (\alpha - \alpha_c)/\alpha_c$

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[9]. Writing $\hat{\mathbf{E}}$ in terms of the complex unit vector $\mathbf{M} = (\mathbf{N} + i\mathbf{B})/\sqrt{2}$, we have

$$\hat{\mathbf{E}} = \alpha(s)\mathbf{M} + \beta(s)\mathbf{M}^*, \qquad (6)$$

with $|\alpha|^2 + |\beta|^2 = 1$.

Substituting this in the Eq. (4), it can be seen that $\alpha(s) = e^{i\chi}\alpha(0)$ and $\beta(s) = e^{-i\chi}\beta(0)$, where

$$\chi = \int_0^S \tau \, ds, \qquad (7)$$

with τ as defined in Eq. (3). For a cyclic evolution, χ can be seen to be equal to the phase increase (decrease) of the right (left) circularly polarized light.[9] Thus the phase change determined solely by the geometry of the cyclic path Γ is fully characterized by its curvature and the torsion. In view of the transverse nature of the electromagnetic waves, $\mathbf{T} \cdot \hat{\mathbf{E}} = 0$. This leads to a very desirable result that the dynamical phase is zero and the net phase change is equal to χ, the geometric phase.

One of the characteristics of the boundary curves of the n-twisted strips is the existence of n inflexion points. At the inflexion point, the curvature vanishes and the tangent to the curve shows a cusp. For the Möbius strip (i.e., $n = 1$), the critical boundary curve has just one inflexion point at $t = 2\pi$ and $\alpha_c = 4/5$. For $n = 3$, the boundary curve is a trefoil, which is knotted, and there are three inflexion points. In general for the boundary curve of an n-twisted strip, n inflexion points occur at a critical width $\alpha_c = 1/(1 + n^2/4)$, when $\cos(nt_c/2) = -1$.[12] As seen in the figure 1, the boundary curves encode the width of the strip and the critical curve, namely the curve with an inflexion point is distinguished by a vanishing of curvature around that point.

Symbolic manipulation facilitates the determination of analytic expressions for the curvature $\kappa(t)$ and torsion $\tau(t)$, by using Eq. (1) in Eq. (3). For the Möbius case, we get

$$\tau = \frac{6\alpha[4\cos\ t/2 + \alpha(5 + 8\cos\ t) + \alpha^2(9\cos\ t/2 + \cos\ 3t/2)]}{\kappa^2},$$

where the curvature κ is given by
$\kappa = 64 + 288\alpha\cos\ t/2 + \alpha^2(284 + 216\cos\ t) + \alpha^3(336\cos\ t/2 + 64\cos\ 3t/2) + \alpha^4(65 + 54\cos\ t + 6\cos\ 2t)$

Taylor expansion of κ and τ near α_c and t_c gives,

$$\kappa \approx \frac{1}{2v_c^{3/2}}[(\alpha - \alpha_c)^2 + 9b^2v_c^2(t - t_c)^2]^{\frac{1}{2}}$$

$$\tau \approx \frac{-3b(\alpha - \alpha_c)}{[(\alpha - \alpha_c)^2 + 9b^2v_c^2(t - t_c)^2]},$$

where $b = \frac{6}{5}$ and $v_c = |d\mathbf{r}/dt|_c = 1/\sqrt{(5)}$. These equations show that τ changes sign as the parameter α passes through its critical value.[7]

Geometrically, the vanishing of κ results in the divergence in the torsion τ at the inflexion point. However, as shown below, this singularity is integrable and the resulting integrated τ, the geometric phase is well defined. To show this, we calculate $\chi(t)$ near the inflexion point. For arbitrary t_0, the total twist in the time interval $(-t_0 + t_c, t_0 + t_c)$ centered around the critical point t_c is, $\int_{t_c-t_0}^{t_c+t_0} \tau v dt = -\int_{t_c-t_0}^{t_c+t_0} \frac{bv_c(\alpha - \alpha_c)dt}{[(\alpha - \alpha_c)^2 + v_c^2b^2(t - t_c)^2]}$. This integral is equal to $2Arc\tan[\frac{t_0}{(bv_c(\alpha - \alpha_c))}]$. As we pass through α_c, it will change from $-\pi$ to π (irrespective of the value of t_0) giving rise to a jump of 2π.

Figure 3 illustrates the evolution of the geometric phase as the light propagates through the boundary curves. For $\alpha < \alpha_c$, the phase factor $\chi(t)$ exhibits a wiggle near the critical point which becomes saw-tooth shaped at criticality. However, for $\alpha > \alpha_c$, the the phase change near the critical points is smooth. As discussed above, and seen in the figure, the polarization of light undergoes a full rotation of 2π after passing through the critical point.

Taylor expansion of the unit normal near the inflexion point shows that as we pass it at $t = t_c$, $\mathbf{N}$ rotates by π about a fixed direction. And for $\alpha < \alpha_c$, it rotates by $-\pi$. The same is true also for $\mathbf{B}$. Therefore, the number of rotations of the pair $\mathbf{N}, \mathbf{B}$ increases by 2π as we pass the inflection point. For an n-twisted strip, the number of rotations is equal to n, for odd n. For even n, there are two boundary curves. The number of rotations for each of them is $(n/2)$. Thus there exists a universal functional form for local geometrical quantities τ and κ and a jump of 2π in the global variables near the inflexion point, for the boundary curves of both orientable as well as non-orientable strips.

Figure 4 shows the net phase accumulated by light after propagating in a cyclic loop along the boundary curve of an n-twisted strip, as its width varies, for $n = 1, 2$ and 3. In figure 5, we plot the phase modulo 2π. A striking
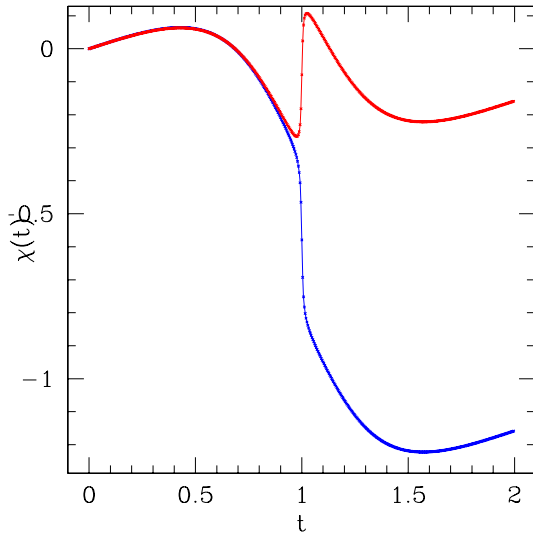
FIG. 3: (color on line) Geometric phase (in units of 2π) acquired by polarized light as it passes through the closed boundary curve of a Möbius strip. The top and the bottom curves correspond respectively to strip widths just below ($\alpha = 0.79$) and just above ($\alpha = 0.81$) the critical width $\alpha_c = 0.8$.
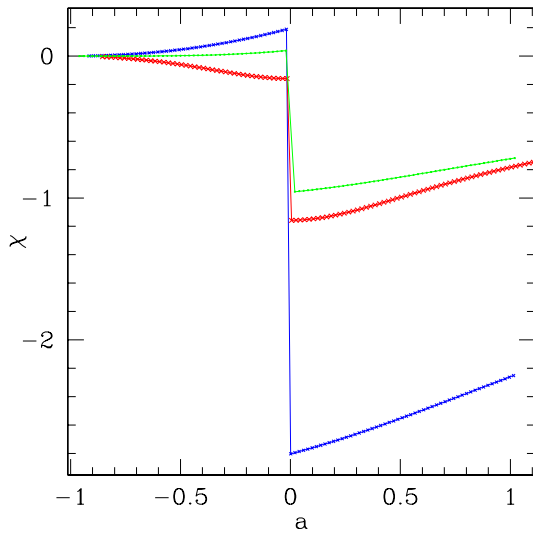

FIG. 4: (color on line) From Top to bottom : Geometric phase (in units of 2π) for the cyclic paths along the boundary curves of a triply-twisted strip ($n = 3$), a doubly twisted strip $n = 2$ and a singly-twisted, i.e., Möbius strip, for various strip widths. Here $a = (\alpha - \alpha_c)/\alpha_c$

FIG. 5: (color on line) Same as Fig. 4, except that the geometric phase is plotted modulo 2π.

aspect is the characteristic minimum that we find in the phase shift in the case of the Möbius circuit characterized by the critical width of the strip. In other words, geometric phase shows that the Möbius geometry with $n = 1$ is indeed special. The absence of such a minimum in the boundary curves of all other (integer) n values makes the origin of such a minimum rather mysterious.
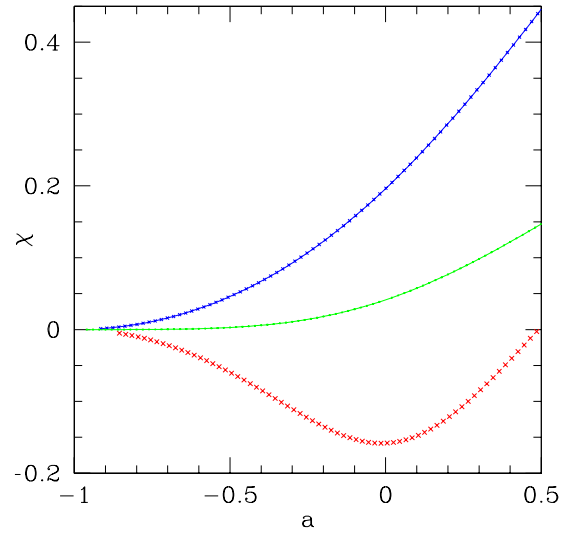
It is rather intriguing that the width for the existence of the diabolic crossing of the eigenspectrum of the quantum Hamiltonian coincides with the critical width resulting in the minimum phase shift. It would have been natural to speculate that the global phase shift is influenced by a local diabolo geometry. However, such a minimum exists only in the Möbius loop, whereas the diabolic crossings characterize the boundary curves of all n-twisted strips. Therefore, the fact that two different and somewhat unique features occur at the same parameter is either a coincidence or it is conceivable that this is happening because the Möbius geometry and topology is unique in the sense that $n = 1$ is the only case where the single boundary curve of the twisted strip is un-knotted and has a single inflexion point. The presence of multiple inflexion points in the n-twisted ribbons (with $n > 1$) seem to play a role in such a way as to wipe out this minimum.

We conclude this paper by suggesting possible experiments that could verify our theoretical results. We begin by noting that in experiments that demonstrate the geometrical phase phenomenon in optics, the usual nonplanar path that has been considered is either an open helix [3], or a helix that can be effectively closed by making the fiber lie along planar paths (with $\tau = 0$) at the two ends of the helix[4]. Clearly, such a circuit has no inflexion points where τ becomes locally singular, since κ is a constant for a helix. To our knowledge, no experiments have been performed on nonplanar fibers with inflexion points. We hope that our theoretical results that show that the presence of inflexion points has a nontrivial effect on the optical phase shift would motivate experimentalists to observe it.

For example, optical fibers which have the shape of the boundary curve of various n-twisted strips can be perhaps be fabricated as follows. By twisting a sheet made of some pliable material such as plastic (or any other material appropriate for optical experiments) of some width α once, and gluing together the two short edges, a closed Möbius strip can be constructed. Instead of winding the optical fiber on a cylinder as was done in the experiments of Tomita-Chiao[3] and Frins-Dultz [4], the fiber could now be attached along the boundary curve of the above Möbius strip, and a similar experimental setup could be used to measure the geometrical phase. The experiment can then be repeated with various different widths of the Möbius, to find the dependence of the geometrical phase on the width, with a characteristic minimum phase oc-curring at a critical width. Similar measurements can be done with optical fibers in the shape of the boundary curve of a strip twisted n times, and repeated for different widths.

Finally, our studies may be relevant in optical fibers used in tuning of polarization, as such a phase can be seen in low birefringent optical fibers[14]. We hope that our studies will also stimulate laboratory research involving lasers and condensates in this novel class of twisted geometries that we have studied.

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